JOURNAL OF APPROXIMATION THEORY **58**, 114–120 (1989)

Uniqueness of Best L_1-Approximations from Periodic Spline Spaces

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Communicated by Oved Shisha

Received December 8, 1987

It is shown that every periodic continuous function has a unique best L_1-approximation from a given periodic spline space, although these spaces are not weak Chebyshev in general. © 1989 Academic Press, Inc.

INTRODUCTION

Standard spaces for approximating periodic continuous functions $f\colon [a, b] \to \mathbf{R}$ (i.e., $f(a) = f(b)$) are spaces of periodic splines. We denote by $P_m(K_n)$ the n-dimensional space of periodic splines of order $m \geqslant 2$ with the set of knots $K_n = \{x_0, ..., x_n\}$, where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

The space $P_m(K_n)$ is weak Chebyshev for odd n. We show that any periodic weak Chebyshev space G (i.e., $g(a) = g(b)$ for all $g \in G$) with some additional property is necessarily of odd dimension. In particular, the space $P_m(K_n)$ is not weak Chebyshev for even n.

Our object is to prove a uniqueness result on best L_1-approximation by periodic splines. The standard spaces for which uniqueness of best L_1-approximations is known are all weak Chebyshev and have even a stronger property (A) (cf. Sommer [4] and Strauss [5]). We show that every periodic continuous function has a unique best L_1-approximation from $P_m(K_n)$, although $P_m(K_n)$ is not weak Chebyshev in general.

0021-9045/89 $3.00

MAIN RESULTS

Let $C^r[a, b]$ be the space of all r-times continuously differentiable real functions on the interval $[a, b]$. The space of polynomials of order at most m is denoted by Π_m. Let a set of knots $K_n = \{x_0, ..., x_n\}$ with $n \geqslant 1$ and $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be given. For $m \geqslant 2$ we call

$$P_m(K_n) = \{s \in C^{m-2}[a, b] : s|_{(x_{i-1}, x_i]} \in \Pi_m, i = 1, ..., n,$$
$$s^{(j)}(a) = s^{(j)}(b), j = 0, 1, ..., m-2\}$$

the space of *periodic splines* of order m with the set of knots K_n.

An n-dimensional subspace G of $C[a, b]$ is called *weak Chebyshev*, if every function $g \in G$ has at most $n-1$ sign changes; i.e., there do not exist points $a \leqslant t_1 < \cdots < t_{n+1} \leqslant b$ such that $g(t_i)\, g(t_{i+1}) < 0$, $i = 1, ..., n$.

We note that by induction on m using Rolle's theorem it is not difficult to verify that every spline in $P_m(K_n)$ has at most $n-1$ (respectively n) sign changes, if n is odd (respectively even). In particular, the n-dimensional space $P_m(K_n)$ is weak Chebyshev for odd n (compare also Schumaker [3]). Our first result on weak Chebyshev spaces of periodic functions implies that this is not true for even n.

A subspace G of $C[a, b]$ is called *periodic*, if $g(a) = g(b)$ for all $g \in G$. This definition differs from that given in Zielke [6, p. 20].

We next show that certain periodic weak Chebyshev spaces must have odd dimension. A similar result, which can be easily derived from Theorem 1, was proved in Zielke [6, p. 20].

THEOREM 1. *Let G be a periodic weak Chebyshev subspace of $C[a, b]$. If there exists a function $g_0 \in G$ with $g_0(a) \neq 0$, then the dimension of G is odd.*

Proof. Let $g_1, ..., g_n$ form a basis of the n-dimensional periodic weak Chebyshev subspace G of $C[a, b]$. Since the functions $g_1, ..., g_n$ are linearly independent, there exist points $a \leqslant t_1 < \cdots < t_n \leqslant b$ such that the determinant $\det(g_i(t_j))_{i,j=1}^n$ is nonzero. Thus there exists a function $g \in G$ such that

$$g(t_i) = (-1)^i, \qquad i = 1, ..., n.$$

We first consider the case $g(a) \neq 0$. Then we have sgn $g(a) = -1$, since otherwise by considering the points a, $t_1, ..., t_n$ we see that g has n sign changes, contradicting the assumption that G is weak Chebyshev. Since g is a periodic function, we have $\operatorname{sgn} g(b) = -1$. For even n we get $\operatorname{sgn} g(t_n) = 1$. By considering the points $t_1, ..., t_n, b$ we see that g has again n sign changes, contradicting our assumption. We now consider the case $g(a) = 0$. Let $g_0 \in G$ be the function with $g_0(a) \neq 0$. We may assume that

$\operatorname{sgn} g_0(a)=1$. For all $\varepsilon>0$ we define the function $g_\varepsilon \in G$ by $g_\varepsilon = g+\varepsilon g_0$. Then

$$\operatorname{sgn} g_\varepsilon(a)=1$$

and for sufficiently small $\varepsilon>0$ we still have

$$\operatorname{sgn} g_\varepsilon(t_i)=(-1)^i, \qquad i=1, ..., n.$$

Hence g_ε has at least n sign changes, contradicting our assumption. This proves Theorem 1.

We note that Theorem 1 is no longer true, if we drop the assumption that there exists a function $g_0 \in G$ with $g_0(a) \neq 0$. This can be seen by following the example.

Let points $a=x_1<x_2<\cdots<x_{n+m}=b$ be given and $G=\operatorname{span}\{B_1^m, ..., B_n^m\}$, where for each $i \in \{1, ..., n\}$ the function B_i^m is the B-spline of order m with support (x_i, x_{i+m}). Then it is well known that G is an n-dimensional periodic weak Chebyshev subspace of $C[a, b]$ such that $g(a)=0$ for all $g \in G$ (see Schumaker [3]).

Following the proof of Theorem 1 we see that the next result holds.

COROLLARY 2. *Let G be a periodic weak Chebyshev subspace of $C[a, b]$ of dimension n. If there exists a function $g_0 \in G$ with $g_0(a) \neq 0$, then there is no function $g \in G$ with $n-1$ sign changes on $[a, b]$ satisfying $g(a)=0$.*

We now investigate the uniqueness of best L_1-approximations from $P_m(K_n)$ for periodic functions in $C[a, b]$.

For all functions $h \in C[a, b]$ the L_1-norm is defined by

$$\|h\|_1=\int_a^b |h(t)|\, dt. \tag{1}$$

Let a subspace G of $C[a, b]$ and a function $f \in C[a, b]$ be given. A function $g_f \in G$ is called a *best L_1-approximation* of f from G, if

$$\|f-g_f\|_1=\inf_{g \in G} \|f-g\|_1. \tag{2}$$

In the following we prove a global unicity result for best L_1-approximations from $P_m(K_n)$. For doing this we need some notations and results.

Given a function $f \in C[a, b]$ we set $Z(f)=\{t \in [a, b]: f(t)=0\}$. Moreover, if A is a subset of $[a, b]$, then we denote by $|A|$ the number of points in A.

The first result on zeros of periodic splines can be found in Schumaker [3].

LEMMA 3. *Let a spline $s \in P_m(K_n)$ be given such that $|Z(s)| < \infty$. If n is even (respectively odd), then $|Z(s) \cap [a, b)| \leqslant n$ (respectively $|Z(s) \cap [a, b)| \leqslant n-1$). Moreover, if $|Z(s) \cap [a, b)| = n$, then s changes sign at the zeros in (a, b).*

The next result on weak Chebyshev spaces is well known (see, e.g., Deutsch, *et al.* [1]).

LEMMA 4. *Let an n-dimensional weak Chebyshev subspace of $C[a, b]$ and points $a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b$ be given, where $0 \leqslant r \leqslant n-1$. Then there exists a nontrivial function $g \in G$ such that*

$$(-1)^i g(t) \geqslant 0, \qquad t \in [t_{i-1}, t_i], \; i = 1, ..., r+1. \tag{3}$$

The following characterization of best L_1-approximations can be found in Rice [2].

THEOREM 5. *Let G be a subspace of $C[a, b]$ and $f \in C[a, b]$. The following statements hold:*

(i) *A function $g_f \in G$ is a best L_1-approximation of f if and only if for all $g \in G$,*

$$\int_a^b g(t) \operatorname{sgn}(f(t) - g_f(t)) \, dt \leqslant \int_{Z(f - g_f)} |g(t)| \, dt. \tag{4}$$

(ii) *If $g_1, g_2 \in G$ are best L_1-approximations of f, then*

$$(f(t) - g_1(t))(f(t) - g_2(t)) \geqslant 0, \qquad t \in [a, b]. \tag{5}$$

We are now in position to prove the announced unicity result.

THEOREM 6. *Every periodic function in $C[a, b]$ has a unique best L_1-approximation from $P_m(K_n)$.*

Proof. Suppose that the claim is false. Then there exists a function $f \in C[a, b]$ such that $s_1 = 0$ and $s_0 \in P_m(K_n)$, $s_0 \neq 0$, are best L_1-approximations of f from $P_m(K_n)$. It follows from Theorem 5 that

$$f(t)(f(t) - s_0(t)) \geqslant 0, \qquad t \in [a, b].$$

This implies that for all $t \in [a, b]$,

$$|f(t) - \tfrac{1}{2}s_0(t)| = |\tfrac{1}{2}(f(t) - s_0(t)) + \tfrac{1}{2}f(t)| = \tfrac{1}{2}|f(t) - s_0(t)| + \tfrac{1}{2}|f(t)|.$$

Therefore, if $f(t)-\frac{1}{2}s_0(t)=0$, then $\frac{1}{2}|f(t)-s_0(t)|+\frac{1}{2}|f(t)|=0$ which implies that $s_0(t)=0$. This shows that

$$Z(f-\tfrac{1}{2}s_0)\subset Z(s_0). \tag{6}$$

Claim. There exists a nontrivial function $s\in P_m(K_n)$ such that

$$(f(t)-\tfrac{1}{2}s_0(t))\,s(t)\geqslant 0, \qquad t\in[a,b], \tag{7}$$

and

$$s(t)=0, \qquad t\in[c,d], \qquad \text{if} \quad f(t)-\tfrac{1}{2}s_0(t)=0,\, t\in[c,d], \tag{8}$$

for all $c<d$.

Suppose for the moment that the claim is true. Then it follows that

$$\int_a^b s(t)\operatorname{sgn}(f(t)-\tfrac{1}{2}s_0(t))>0=\int_{Z(f-(1/2)s_0)}|s(t)|\,dt.$$

Then by Theorem 5 the spline $\frac{1}{2}s_0$ is not a best L_1-approximation of f from $P_m(K_n)$ which is a contradiction, since $s_1=0$ and $s_0\in P_m(K_n)$ are best L_1-approximations of f. Therefore, it remains to prove the existence of the spline s as in the claim. It suffices to consider three cases.

Case 1. $|Z(s_0)|<\infty$. We first consider the case when n is odd. It follows from Lemma 3 that $|Z(s_0)\cap(a,b)|\leqslant n-1$. Then by (6) the function $f-\frac{1}{2}s_0$ has at most $n-1$ sign changes. Thus there exists a sign $\sigma\in\{-1,1\}$ and points $a=t_0<t_1<\cdots<t_r<t_{r+1}=b$, where $0\leqslant r\leqslant n-1$, such that

$$\sigma(-1)^i\,(f(t)-\tfrac{1}{2}s_0(t))\geqslant 0, \qquad t\in[t_{i-1},t_i],\, i=1,\dots,r. \tag{9}$$

Since n is odd, $P_m(K_n)$ is an n-dimensional weak Chebyshev space. Therefore, by Lemma 4 there exists a nontrivial functon $s\in P_m(K_n)$ such that

$$\sigma(-1)^i s(t)\geqslant 0, \qquad t\in[t_{i-1},t_i],\, i=1,\dots,r. \tag{10}$$

Then it follows from (9) and (10) that the spline s has the desired property (7).

We now consider the case when n is even. We set $K_{n-1}=\{y_0,\dots,y_{n-1}\}$, where $y_i=x_i$, $i=0,\dots,n-2$, and $y_{n-1}=b$. Since $n-1$ is odd, $P_m(K_{n-1})$ is an $(n-1)$-dimensional weak Chebyshev space.

Case 1.1. $f(a)-\frac{1}{2}s_0(a)=0$. It follows from (6) that $s_0(a)=0$. Then by Lemma 3 we have $|Z(s_0)\cap(a,b)|\leqslant n-1$. Therefore, by (6) the function $f-\frac{1}{2}s_0$ has at most $n-1$ sign changes. If $f-\frac{1}{2}s_0$ has at most $n-2$ sign changes, then analogously as in the case when n is even, there exists a spline $s \in P_m(K_{n-1}) \subset P_m(K_n)$ satisfying (7). If $f - \frac{1}{2}s_0$ changes sign at $n-1$ points $t_1 < \cdots < t_{n-1}$ in (a, b), then by (6) we have $t_1, ..., t_{n-1} \in Z(s_0)$. Since $s_0(a) = 0$, it follows from Lemma 3 that $Z(s_0) \cap (a, b) = \{t_1, ..., t_{n-1}\}$ and s_0 changes sign at the points $t_1, ..., t_{n-1}$. Therefore, the spline $s = s_0$ or $s = -s_0$ satisfies (7).

Case 1.2. $f(a) - \frac{1}{2}s_0(a) \neq 0$. It follows from Lemma 3 that $|Z(s_0) \cap (a, b)| \leqslant n$. Then by (6) we have $|Z(f - \frac{1}{2}s_0) \cap (a, b)| \leqslant n$. Moreover, since $f(a) - \frac{1}{2}s_0(a) = f(b) - \frac{1}{2}s_0(b) \neq 0$, the function $f - \frac{1}{2}s_0$ has an even number of sign changes. If $f - \frac{1}{2}s_0$ has at most $n-2$ sign changes, then analogously as in Case 1.1 there exists a spline $s \in P_m(K_{n-1}) \subset P_m(K_n)$ satisfying (7). If $f - \frac{1}{2}s_0$ changes sign at n points $t_1 < \cdots < t_n$ in (a, b), then by (6) we have $t_1, ..., t_n \in Z(s_0)$. Moreover, it follows from Lemma 3 that $Z(s_0) \cap (a, b) = \{t_1, ..., t_n\}$ and s_0 changes sign at the points $t_1, ..., t_n$. Therefore, the spline $s = s_0$ or $s = -s_0$ satisfies (7).

Case 2. $s_0(t) = 0$, $t \in [x_k, x_l] \cup [x_p, x_q]$, where $k < l < p < q$, and $|Z(s_0) \cap (x_l, x_p)| < \infty$. It is well known that

$$G = \{s|_{[x_k, x_q]} : s \in P_m(K_n) \text{ and } s(t) = 0, t \in [x_k, x_l] \cup [x_p, x_q]\}$$

is a $(p-l-m+1)$-dimensional weak Chebyshev space. Since $s_0|_{[x_k, x_q]} \in G$ and $|Z(s_0) \cap (x_l, x_p)| < \infty$, we have $|Z(s_0) \cap (x_l, x_p)| \leqslant p-l-m$ (see Schumaker [3]). Then by (6) the function $f - \frac{1}{2}s_0$ has at most $p-1-m$ sign changes in (x_l, x_q). Therefore, analogously as above there exists a spline $s \in G$ such that

$$(f(t) - \tfrac{1}{2}s_0(t))\, s(t) \geqslant 0, \qquad t \in [x_l, x_p].$$

We now extend s to $[a, b]$ by defining

$$s(t) = 0, \qquad t \in [a, x_k] \cup [x_q, b],$$

which implies that $s \in P_m(K_n)$ has the desired properties (7) and (8).

Case 3. $s_0(t) = 0, t \in [x_p, x_q]$, where $p < q$, and $|Z(s_0) \cap ([a, b] \setminus [x_p, x_q])| < \infty$. By identifying b with a we may consider the interval $[a, b)$ as a circle T with circumference $b - a$. We set

$$y_i = x_{i+q}, \qquad i = 0, ..., n-q,$$

and

$$y_i = x_{i-n+q}, \qquad i = n-q+1, ..., n-q+p.$$

Then the space

$$\{s \in P_m(K_n) : s(t) = 0, t \in [x_p, x_q]\}$$

may be identified with the space

$$H = \{s \in C^{m-2}(T) : s|_{[y_{i-1}, y_i]} \in \Pi_m,\ i = 1, ..., n-q+p,$$
$$\text{and} \quad s(t) = 0,\ t \in [y_{n-q+p}, y_0]\}.$$

The space H may be considered as a usual spline space and it is well known that H is a $(n+p-q-m+1)$-dimensional weak Chebyshev space. Since $s_0 \in H$ and $|Z(s_0) \cap ([a, b] \setminus [x_p, x_q])| < \infty$ we have $|Z(s_0) \cap ([a, b] \setminus [x_p, x_q])| \leqslant n+p-q-m$ (see Schumaker [3]). Then by (6) the function $f - \frac{1}{2}s_0$ has at most $n+p-q-m$ sign changes in $[a, b] \setminus [x_p, x_q]$. Therefore, analogously as above there exists a spline $s \in H$ satisfying (7) and (8). This proves Theorem 6.

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Printed in Belgium